On the theory of the wind-driven ocean circulation

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(Received 14 June 1961)

A surface distribution of stress is imposed on an ocean enclosed by two continental boundaries; the resulting transport circulation is studied between two latitudes of zero surface wind-stress curl, within which the curl reaches a single maximum. Under the assumption that turbulent transfer of relative vorticity has a minimum effect on the mean circulation, inviscid flow patterns are deduced in the limit of small transport Rossby number. Inertial currents, or naturally scaled regions of high relative vorticity, occur on both the eastern and the western continental coasts. Limits on the relative transports of the currents are obtained and found to depend on the direction of variation of the wind-stress curl with latitude, relative to that of the Coriolis accelerations. The most striking feature of the inviscid flow is a narrow inertial current the axis of which lies along the latitude of maximum wind-stress curl. All eastward flow occurs in this mid-latitude jet.

A feature of the flow which cannot remain essentially free of turbulent processes is the integrated vorticity relationship, since the imposed wind-stress distribution acts as a net source of vorticity for the ocean. Heuristic arguments are used together with this integral constraint to deduce the presence and strength of the turbulent diffusion which must occur in the region of the mid-latitude jet. It is further inferred that the turbulent meanders of the jet must effect a net meridional transport of relative vorticity.

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1. Introduction

Considerable progress has been made in the last fourteen years towards an understanding of long-time average, large-scale ocean currents. If the sufficiently complex problem of the general ocean circulation be separated from the more fundamental problem of the circulation of the coupled atmosphere and ocean, the requirement of continuity of (turbulent) stress at the interface of the two fluids becomes the primary driving force acting upon the sea. That is, motions induced by surface wind-stress dominate those caused by the pressure gradient normal to the gravitational field caused, in turn, by the differential heating due to solar radiation. These two driving mechanisms do, of course, interact in an inherently non-linear fashion, and any separation is, to some extent, arbitrary. However, if all motion vanishes at some depth at which the internal tangential stress is also negligible, the total local horizontal transport can be related unambiguously to the surface wind-stress curl over the major part of the oceans (Sverdrup 1947). Although the assumptions inherent in the above statement are not strictly valid (Stommel 1958, chapter 11), the extremely simplified problem which they pose for the transport fields contains features which must be inherent in any more realistic model. Since those aspects of the general circulations to be discussed in this paper can be presented most simply under the assumptions of vanishing motion and stress at some constant depth, this model will be adopted here and refinements directed towards a more realistic ocean model will be left for future discussion.

Furthermore, since the quasi-geostrophic north-south transport directly forced by the wind-stress curl vanishes at certain latitudes, it has become customary to consider separately the circulation of a region bounded by two such latitudes and by two continents, e.g. the Pacific between 13° and 50° N. This point will be discussed explicitly below. In such a region, the wind-stress curl is of one sign and reaches a maximum at approximately the middle latitude, e.g. the curl reaches a maximum at about 33° N. The simplest realistic model of surface wind stress acting in such a region may be taken to have a component only in the longitudinal direction, and the longitudinal component may be taken as a function of latitude alone. This model is developed below.

The paper is presented in six sections, with some small amount of repetitive discussion, so that the more geophysically oriented sections, 2 and 3, and the more mathematically oriented sections, 4 and 5, may each be reasonably self-contained. It is clear, however, that the theory proferred depends critically on the arguments presented in all sections. For convenience the boundary-layer notation used differs somewhat in the two parts.

2. The transport theory of the general circulation

2.1. *Formulation*

To treat the simplest ocean model of this type, we consider, on the β-plane,† an ocean bounded by two latitudes at which the curl of the wind stress vanishes and by two rigid boundaries at constant longitude (meridional continents). Integrating the horizontal momentum equations and the equation of continuity between a constant level of no motion and an undistorted upper sea surface, we have

$$-\int_{-H}^{0} F_1(x,y,z)\,dz+\int_{-H}^{0}(uu_x+vu_y+wu_z)\,dz-2\Omega f(y)\,V+\frac{1}{\rho}P_x=\tau(y), \tag{2.1}$$

$$-\int_{-H}^{0} F_2(x,y,z)\,dz+\int_{-H}^{0}(uv_x+vv_y+wv_z)\,dz+2\Omega f(y)\,U+\frac{1}{\rho}P_y=0, \tag{2.2}$$

$$U_x+V_y=0, \tag{2.3}$$

where subscripts indicate partial differentiation.

† A system of rotating Cartesian co-ordinates. The rotation vector is vertical and has a variable magnitude in the latitudinal direction, thus modelling the radial component of the earth's rotation. The horizontal component is neglected, as the resulting Coriolis accelerations are relatively unimportant. To derive from spherical co-ordinates the β-plane approximation we employ, it must be assumed that the tangent of the latitude is a small quantity; thus the mean latitude of our ocean must be less than 45°. Ultimately, however, we regard the β-plane as a model system.

The following nomenclature is employed:

(x, y, z): co-ordinates in the longitudinal, latitudinal and vertical directions
(u, v, w): the corresponding velocity components
p: the pressure
ρ: the density
H: the depth below the sea surface, $z = 0$, at which motion and tangential stress are assumed to vanish
Ω: magnitude of the earth's rotation
$f(y)$: non-dimensional Coriolis parameter, i.e. $f(y) = f_0 + \beta y/b$, where $f_0 = \sin\theta_0$, $\beta = b\cos\theta_0/R$, and θ_0 is the mean latitude, b the latitudinal extent of the ocean basin, R the radius of the earth
$F_1(x, y, z)$, $F_2(x, y, z)$: components of horizontal frictional force per unit mass (lateral turbulent stresses)
$\tau(y)$: the longitudinal component of surface wind stress
$U \equiv \int_{-H}^{0} u\,dz$, $V \equiv \int_{-H}^{0} v\,dz$, $P \equiv \int_{-H}^{0} p\,dz$: the horizontal transport components and integrated pressure function

In the equations considered, ρ has been treated as a constant, an assumption compatible with the Boussinesq approximation. This, of course, is not the same as the assumption of barotropy. Note that, if the first two terms appearing on the left-hand sides of each of equations (2.1), (2.2) can be adequately represented in terms of the transport fields and their derivatives, a closed problem for the horizontal transports is formed by these three equations alone. Under such representation, these equations will form the basis of the present study, as they have for the previous studies which will first be discussed below.

A useful relationship, the vorticity equation, is obtained upon elimination of the pressure between (2.1) and (2.2),

$$\int_{-H}^{0} (F_{1y} - F_{2x})\,dz + \int_{-H}^{0} [-(uu_x + vu_y + wu_z)_y + (uv_x + vv_y + wv_z)_x]\,dz + 2\Omega\beta\,V/b = -\tau'(y). \quad (2.4)$$

No term proportional to $f(y)$ appears in the vorticity equation (2.4) because of the divergence relation (2.3). This results in the particularly significant dynamical role of the variation of effective Coriolis acceleration with latitude. For, where friction is negligible and the motion is slow enough for the neglect of non-linear terms, equation (2.4) becomes a balance between two terms only, viz.

$$2\Omega\beta V/b = -\tau'(y).$$

This is equivalent to assuming that, except for the component of internal stress necessary to transmit the surface driving force to the body of the fluid, the motion is geostrophic.

2.2. *Quasi-geostrophy*

That oceanic flow is essentially geostrophic is empirically well-known, and the frictionless, linear vorticity equation, together with mass continuity, represents the original quasi-geostrophic† model for the theoretical consideration of the transport fields developed by Sverdrup (1947). The divergence equation (2.3) may be used to define a transport stream function Ψ, in terms of which this model is represented mathematically by a single exceedingly simple equation. Let

$$U = -\Psi_y, \quad V = \Psi_x. \tag{2.5}$$

Then

$$\Psi_x = -(b/2\Omega\beta)\,\tau'(y), \tag{2.6}$$

which has the solution

$$\Psi = -(b/2\Omega\beta)\,\tau'(y)\,x + k(y), \tag{2.7}$$

where $k(y)$ is an arbitrary function of integration. Note that the latitudinal transport is completely specified by equation (2.6), and is unidirectional if $\tau'(y)$ is of one sign. On the other hand, the longitudinal transport is completely unspecified without a consideration of boundary conditions, i.e. without the determination of $k(y)$. Recall that this occurs despite the fact that the surface wind stress is purely longitudinal. It is the cross-wind component of transport which is determined and only the cross-wind component.

Since the directly forced latitudinal flow vanishes at latitudes of zero gradient of surface stress in virtue of equation (2.6), the region between two such latitudes may be considered separately from the rest of the world-ocean. This is assuming, of course, that no other mechanism induces a non-zero transport distribution along the bounding latitude circles. We shall proceed under this assumption as has been done by previous authors. Note, however, that care must be exercised when the results of such a model are applied to a discussion of the circulation of the real oceans. We return briefly to this point in §2.5 below. For simplicity we consider a rectangular ocean $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, subject to the wind-stress

$$\tau(y) = -\frac{\tau_0}{\pi}\cos\frac{\pi y}{b}. \tag{2.8}$$

Thus $\tau'(y) = (\tau_0/b)\sin(\pi y/b)$ (see figure 1). The value of the stream function on the bounding latitudes may be taken as zero. To complete the description of the circulation between them, it remains only to specify that the zero stream line also lies along the eastern and western continental boundaries. However, due to the appearance of only one integration function in the solution (equation (2.7)), this is impossible. The stream function may be made zero at only one longitude. The quasi-geostrophic model is thus degenerate, in the sense that it is incapable of describing the closed circulation of an isolated ocean. This degeneracy may be recognized alternatively by noting the net transport of mass obtained upon integrating V between $x = 0$ and $x = a$ along any constant $y \neq 0$ or b. Sverdrup (1947), considering the detailed structure of equatorial currents in the eastern Pacific (in terms of a more realistic wind-stress representation), determined $k(y)$ by satisfying the boundary condition along the east-coast at $x = a$. This corresponds to a choice of $k(y) = (ab/2\Omega\beta)\,\tau'(y)$, whence

$$\Psi(x,y) = (b/2\Omega\beta)\,\tau'(y)\,(a-x). \tag{2.9}$$

† The term quasi-geostrophic is used throughout this paper precisely as defined here, n.b. *not* as commonly used in meteorology.

Empirical motivation for this choice lies in the fact that at low latitudes strong currents form and flow along the western boundaries, e.g. the Kuroshio. It may be anticipated, therefore, that, at low latitudes, the region near the western coast is dynamically more complicated than that near the eastern coast.

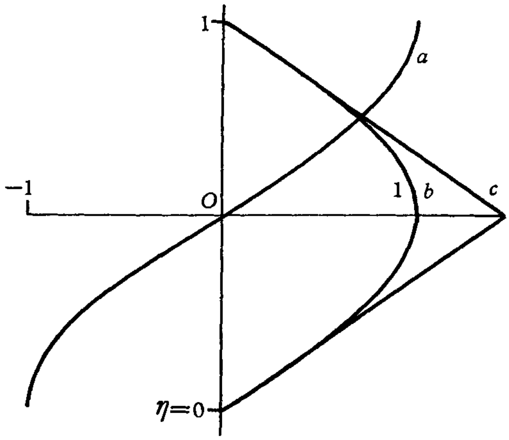

FIGURE 1. (*a*) Sinusoidal form of a simple horizontal wind stress; (*b*) the corresponding latitudinal gradient (wind-stress curl); (*c*) the triangular approximation to the curl as employed in § 3.

The quasi-geostrophic degeneracy must be removed by including at least some of the terms neglected in the vorticity equation (2.4), which, in its complete form, is certainly capable of describing a closed ocean circulation. The terms to be included will still remain small over most of the area of the ocean, but will become controlling in certain limited regions, e.g. boundary layers near coasts. Thus the simple balance given by equation (2.6) will still obtain almost everywhere. It is important to note, however, that since $k(y)$ is determined only by a consideration of the singular regions, the processes of friction and/or inertia (which are directly important only in limited regions) have a gross effect on the flow everywhere. Only after the details of the singular regions have been considered is the east-west transport, $U = b(2\Omega\beta)^{-1}\tau''(y)x - k'(y)$, known in the quasi-geostrophic region. It is found that $k(y)$ depends strongly on the assumptions made to relate the integrated frictional and inertial terms to the mean transport fields.

2.3. *The diffusion of relative vorticity*

The first models which yielded complete solutions for closed oceanic vortices were made under the assumption that the inertial terms (the non-linear coupling of the mean field with itself) remained negligible. The system was closed by the inclusion of horizontal turbulent friction (the non-linear coupling of the zero-average fluctuation fields). This was originally done most simply under the assumption of a frictional force proportional to the horizontal transport velocity (Stommel 1948), and later developed under the assumption of a constant horizontal eddy coefficient (Munk 1950). The mathematical statements are, respectively,

$$\int_{-H}^{0} F_1 dz = RU, \quad \int_{-H}^{0} F_2 dz = RV \quad \text{and} \quad \int_{-H}^{0} F_1 dz = \nu\nabla^2 U, \quad \int_{-H}^{0} F_2 dz = \nu\nabla^2 V,$$

where R and ν are free parameters, adjusted, in each case, to give the best fit to observations. The equations governing the two models are

$$R\nabla^2\Psi + 2\Omega\beta\Psi_x/b = -\tau'(y)$$

and

$$\nu\nabla^4\Psi + 2\Omega\beta\Psi_x/b = -\tau'(y).$$

Both equations are linear and tractable. For the trigonometric wind distribution given by (2.8), Ψ may be taken proportional to sin $(\pi y/b)$ and the equations separated.† If R and ν be assumed small, the resulting x-equations are simply soluble by the technique of singular perturbation theory (Munk & Carrier 1950). To apply this technique, the ocean is initially separated into three regions, an interior and a boundary-layer region near each coast, i.e. at $x = 0, a$. The appropriate interior approximation is that of quasi-geostrophy, and the interior solution is given again by (2.7). In the boundary-layer regions the flow is approximately free from the direct (local) wind stress, a balance being obtained in the vorticity equation primarily between frictional and variation-of-Coriolis-parameter terms. Formally joining the boundary-layer solutions to the interior, the solution is completed and $k(y)$ is unambiguously determined. It is deduced for both models that, in a formal first approximation, the contribution from the boundary layer along the eastern coast vanishes identically. Thus $k(y)$ is determined from the interior solution satisfying by itself the condition $\Psi = 0$ at $x = a$, and the quasi-geostrophic interior is again given by (2.9). The streamline pattern for the asymmetric vortex is sketched in figure 2b. Mathematically the difference between the eastern and western coastal regions results from the fact that the boundary-layer equations consist of a balance between an even and an odd x-derivative. This results in a single sign change between the eastern and western regions when the equations are expressed in terms of local longitudinal variables positive in the direction of outward normals from the coasts (the boundary-layer variables).

In terms of the frictional models, a complete theoretical description of a general ocean circulation was for the first time achieved. Although not at all satisfactorily treating the turbulent process involved, the constant-eddy-viscosity model was considered the more plausible one, and was developed in some detail, including a realistic treatment of wind-stress distributions and ocean-basin shapes. As in the simple model discussed above, the interior solution satisfies alone the eastern boundary condition. An important consequence is that at each latitude the transport of the western boundary current is completely specified independently of the eddy viscosity, and is given by $\psi(0,y)$ of equation (2.9). The width of the western boundary current does, however, depend on the eddy viscosity, and is appropriately measured by the length $(\nu a/2\Omega)^{\frac{1}{3}}$. The eddy viscosity is then determined by making this length scale agree with observation.

The frictional ocean model described above does yield a westward oriented asymmetrical vortex as the response to a simple wind stress, but as an acceptable

† The higher order of the differential equation for the model with constant eddy viscosity implies that additional boundary conditions have to be specified. These are taken as $\partial\Psi/\partial x = 0$, $x = 0, a$; $\partial^2\Psi/\partial y^2 = 0$, $y = 0, b$; i.e. rigid, and tangential-stress-free surfaces respectively. The separation of Ψ is seen to remain valid.

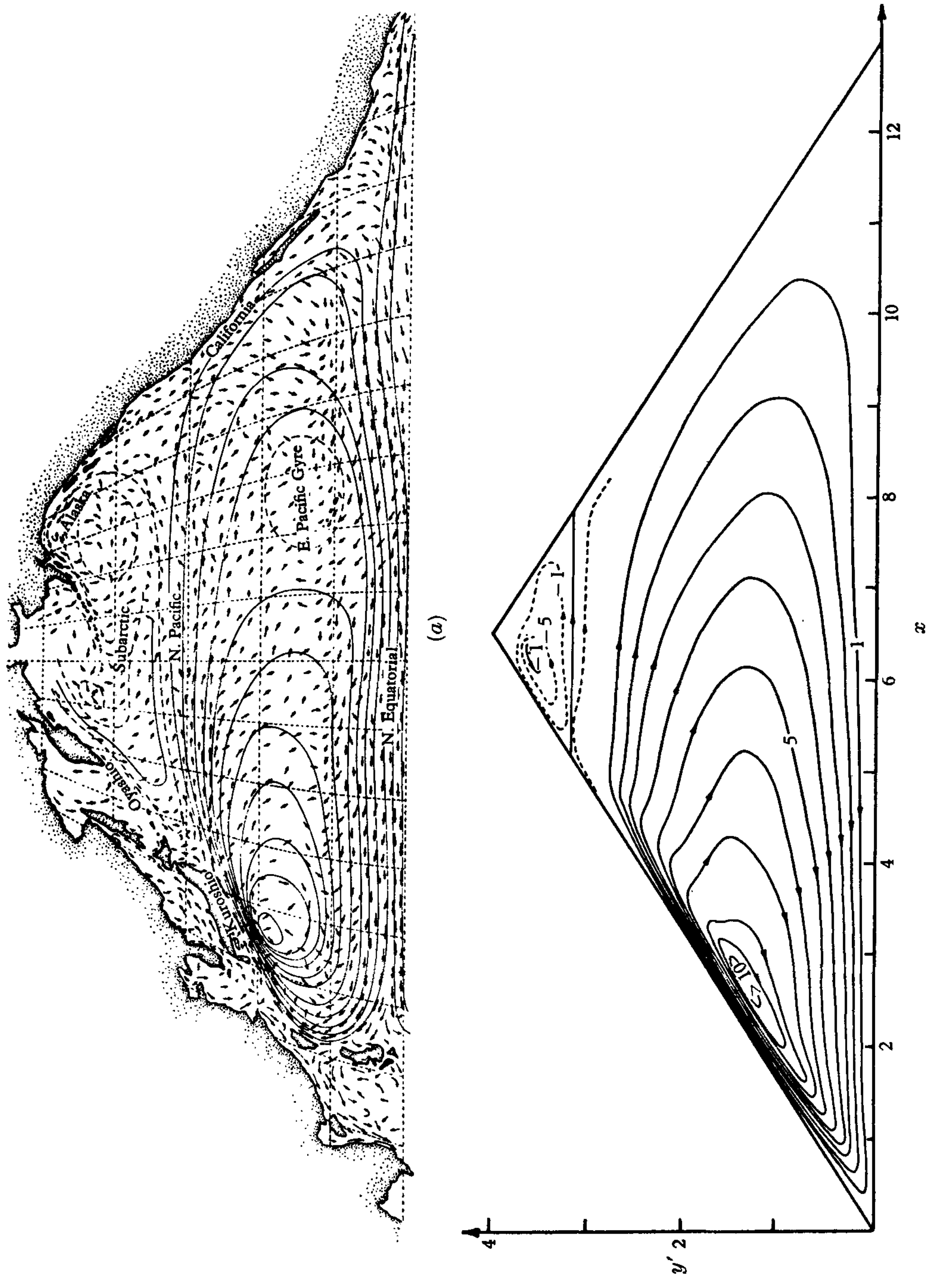

FIGURE 2. (*a*) Transport streamlines derived from observations on the North Pacific ocean; (*b*) transport streamlines as given by the frictional theory applied to a triangular ocean with an eddy viscosity of 10^8 cm²/sec. (From Munk & Carrier 1950.)

theory of the general circulation in the real oceans it is open to several criticisms. First, the eddy viscosity required to give the observed width of the Gulf Stream to the theoretical boundary current is significantly greater than the value of the eddy viscosity indicated in the Gulf Stream region by independent means. Secondly, the theoretically deduced transports of the western boundary currents are much smaller than observed transports, smaller by a factor of 0·5 for the Gulf Stream and 0·6 for the Kuroshio (Munk 1950, table 2). Thirdly, the predicted streamline pattern is in qualitative disagreement with observations. This may be seen from figure 2, which compares the theory applied to the North Pacific Ocean with the observed transport field. Since the only strong current predicted on the frictional theory flows northward along the entire length of the western boundary, the observed Kuroshio may be accounted for below about 35°. The theory cannot, however, explain the Kuroshio leaving the coast at this latitude and, holding together, flowing eastward out to sea; nor does it account for the California or the Alaska currents. It has been speculated that the Kuroshio leaves the western coast because of an instability of the simple flow given by the frictional theory. The circulation which includes the Alaska current has been considered to be an independent vortex, which appears in the apex of the triangular ocean of figure 2*b*. It may be seen that the theoretical gyre begins at too high a latitude, is too small, and has only a western boundary current.

2.4. *The advection of relative vorticity*

The above discrepancies preclude the possibility of a purely frictional theory of the general ocean circulation. This possibility is precluded even if one is seeking only an approximate explanation of the grossest features of the motions which occur in nature. This means that the boundary-layer control of the general ocean circulation cannot be dominated by the frictional diffusion of vorticity. It must, therefore, be dominated by the non-linear process of vorticity advection. Initial work on the development of such an inertial theory (Charney 1955; Morgan 1956) seems to have been motivated only by the discrepancy between the theoretically required eddy-viscosity and the observational upper bound,† under the stimulation of the results of a very simple inertial model (the conservation of potential vorticity in a current independent of latitude, see Stommel 1958, p. 109) which provided a suggestively accurate description of a western boundary current. An immediate advantage of a thoery in which the inertial terms completely dominate the frictional ones is that no free parameter, characterizing the turbulence, is present. It is possible to subject the theory to stringent comparison with nature.

Due to the non-linearity of the inertial terms, it is not possible to develop in a straightforward manner a theory for the transport fields alone. We shall

† If one accepts 10^6 cm²/sec obtained from Pillsbury's measurements in the Florida Straits (Stommel 1955) as characteristic of the maximum turbulence present in the boundary-current region, then the 10^8 cm²/sec required by the frictional theory is larger than that which is available. If, on the other hand, one is willing to accept a narrower stream from the theory and uses 10^6 cm²/sec, the inertial terms are of comparable magnitude to the viscous terms. The frictional theory as developed is thus not self-consistent, and it becomes reasonable to explore next the restriction $\nu \ll 10^6$.

proceed to do so, however, by evaluating the integrated inertial terms by means of an assumption; thus we replace equation (2.4) by

$$H^{-1}[-(UU_x+VU_y)_y+(UV_x+VV_y)_x]+2\Omega\beta V/b+\tau'(y)=0. \tag{2.10}$$

In other words, we replace the actual problem of interest, that of a three-dimensional flow driven by a surface force, by an analogous problem of a two-dimensional flow driven by a body force. The approximate evaluation of the advective integrals inherent in the analogy is, of course, valid only when the vertical velocity and the vertical variation of the horizontal velocity are negligible. These conditions certainly are not fulfilled over the major part of the ocean in the presence of the wind-driven surface Ekman layer and the corresponding convergence or divergence. But over the major part of the ocean, the inertial terms will be anyhow negligible, becoming important only in intense and narrow currents. These streams of high relative vorticity are not driven primarily by the local winds (divergence of the local Ekman layer), but by a horizontal flux of mass into the region of the intense current. This horizontal mass flux has originated from the effect of the winds blowing over the whole ocean basin. Furthermore, the downstream component of flow in the narrow current remains approximately geostrophic. Under these conditions, equation (2.10) provides an appropriate approximation everywhere for the upper layer of a two-layer theory. Although it is necessary in a proper two-layer theory to allow for a variation in depth of the upper layer, i.e. to let $H = H(x, y)$, we shall treat only the case of constant H. In terms of the understanding of the relationship between the quasi-geostrophic regions and the streams of high relative vorticity provided by this simple example, a more sophisticated model may be evolved.

In the previous theories mentioned above, the depth of the upper layer was treated as variable. The studies were not, however, concerned with a complete inertial theory in the sense of the determination of $k(y)$ and the associated quasi-geostrophic flow by a simultaneous consideration of boundary-layer and interior regions. They were concerned rather with the investigation of particular features of inertial boundary currents with a given interior flow. Furthermore, consideration was given only to the flow in the equatorial half of an ocean basin, the region below the maximum of the wind-stress curl. It will be seen below that results so obtained are not in general valid over the entire basin.

Both studies were influenced by the fact that in previous ocean models the interior stream function satisfied by itself the eastern coast boundary condition. Charney assumes that the interior solution should in fact be given by the Sverdrup–Munk transport function, but noting that the transport prescribed into the Gulf Stream region is too low on this theory, replaces it by the observed transport function at the Gulf Stream edge. With this empirical interior Charney computed by numerical integration the structure of a boundary current in a two-layer inertial model which allowed for variation in depth of the upper layer. He found good agreement with the observed structure of the Gulf Stream. The good agreement obtained by a proper local theory provides some justification for our cruder treatment of the inertial terms when we consider the complete problem.

Morgan considered a greater range of particular types of inertial boundary layers. None were as directly applicable to a real oceanic situation as was Charney's study, but were coupled with theoretically deduced interior solutions. To simplify the non-linear analysis, the wind-stress curl was approximated by a linear function away from the lowest latitude. For a constant layer-depth Morgan considered two choices of interior solution, the Sverdrup–Munk solution satisfying the eastern coast boundary condition, $k(y) = (ab/2\Omega\beta)\tau'(y)$, and a solution which satisfied the western coast boundary condition, $k(y) = 0$. For the first case the flow could be closed by an inertial boundary current, for the second it could not. Then, retaining the satisfaction of the eastern condition by the interior function, the effects of density stratification in terms of a variable layer-depth were investigated. The existence and width of the boundary current were not markedly altered by the variable depth. The transport of the western boundary current is, of course, independent of whether or not H is varied when the interior function is assumed to satisfy the east coast conditions. Under this assumption, the interior function is the same as that deduced on the frictional theory, and the transport discrepancy is assumed in Morgan's model.

Before proceeding to develop an inertial theory in which the interior and boundary-layer solutions are treated in full generality, we shall first explore more fully the case of linear wind-stress curl as posed by Morgan. The analysis will remain quite straightforward and the results will exemplify the features of greatest interest of the complete inertial theory. A more general interior solution (which contains Morgan's two solutions as special cases) will be used. Morgan's solution for an equatorial half-basin will be shown to be the end-point of a class of possible solutions and it will be shown that Morgan's choice corresponds to that of minimum transport in the western boundary current. Considering similarly the poleward half of an ocean basin, significantly different constraints on the class of interior solutions allowed will be obtained. Combining these results to infer the flow over a complete ocean basin, the most striking feature is the existence of a strong and narrow eastward flowing current at the latitude of maximum wind-stress curl.

The determination of $k(y)$ is, although in a highly non-linear fashion, related to the solutions of equation (2.10) when the forcing inhomogeneity, $\tau'(y)$, vanishes identically. Such free inertial flow has been considered by Fofonoff (1954). Although we shall not make direct use of the free solutions in our following development, in the free solutions geostrophic regions of high relative vorticity are related to one another in a general way which is characteristic of the forced problem. A discussion of the free problem is presented in the next section.

2.5. *Free inertial flow*

We consider here solutions of equation (2.9) for the case of $\tau'(y) \equiv 0$. Introducing the stream function as defined by (2.5), the terms may be arranged in the form

$$\Psi_x\{\nabla^2\Psi + 2\Omega hf(y)\}_y - \Psi_y\{\nabla^2\Psi + 2\Omega hf(y)\}_x = 0. \qquad (2.11)$$

Simple integration gives the first integral in the form

$$\nabla^2\Psi + 2\Omega hf = G(\Psi). \qquad (2.12)$$

As was done by Fofonoff, we shall investigate only the class of free solutions for which (2.12) becomes linear, i.e. we investigate the case of

$$G(\Psi) = g_0 + g_1\Psi, \tag{2.13}$$

where g_0 and g_1 are numerical constants (a velocity and an inverse squared length respectively, recalling that Ψ is a transport stream function). We seek conditions under which the free solution will contain a geostrophic region and regions of high relative vorticity, i.e. inertial currents, retaining the conditions that Ψ vanish on all sides of a bounding rectangle.

Let the stream-function be non-dimensionalized by its maximum value Ψ_0, which is of course indeterminate for a free solution. We introduce also non-dimensional longitude and latitude variables; by substitution of (2.13), (2.12) becomes

$$\alpha\gamma_1(\lambda^2\phi_{\xi\xi} + \phi_{\eta\eta}) - \gamma_1\phi - \gamma_0 + \eta = 0, \tag{2.14}$$

where $\Psi = \Psi_0\phi, \quad x = a\xi, \quad y = b\eta,$

and $\gamma_0 = (f_0 - g_0 h^{-1})/\beta, \quad \gamma_1 = g_1\Psi_0/2\Omega\beta h, \quad \alpha = g_1/b^2, \quad \lambda = a/b.$

The conditions under which a geostrophic region and singular regions of narrow inertial currents can exist may now be extracted. For succinctness we shall exclude extreme geometry, i.e. we assume the length to width ratio of the ocean basin, λ, to be of order unity. Then a geostrophic region will exist if $|\alpha| \ll 1$, the approximate solution to (2.14) being given by

$$\phi_g = (\eta - \gamma_0)/\gamma_1. \tag{2.15}$$

Note that the geostrophic stream function is zero only on $\eta = \gamma_0$, so that at least three inertial boundary layers are required to yield a complete free solution. The non-dimensional width of the inertial boundary layers will be $O(\alpha^{\frac{1}{2}})$, irrespective of whether the layers are near bounding latitudes or longitude. The condition that the resulting approximate equation describe a boundary-layer phenomenon, i.e. a narrow current, is that its solution contains a decaying real exponential. This condition is seen to be $\alpha > 0$. A solution with a natural length scale $O(|\alpha|^{\frac{1}{2}})$ but for which $\alpha < 0$ is indeed possible; it would have the form of a rapidly oscillating inertial wave existing over the entire ocean basin. Such solutions are certainly of interest but will not be discussed further here.

In summary, note that both conditions obtained are in the nature of restrictions upon α, and may be expressed as

$$0 < \alpha = g_1 b^{-2} \ll 1. \tag{2.16}$$

Thus the existence of a geostrophic region which can be closed by inertial boundary layers, and the characterization of these boundary layers, depends only upon the integration function $G(\Psi)$. In particular, there is no fundamental dependence upon β, the variation of the Coriolis parameter with latitude. This fact is not clearly stated by Fofonoff. What does depend on β, however, is the nature of the geostrophic interior.

From (2.15) the geostrophic velocities are expressed as

$$U_g = -2\Omega\beta h/g_1, \quad V_g = 0. \tag{2.17}$$

Thus, if $\beta = 0$, the motion is confined entirely to the boundary layers. If $\beta \neq 0$, from (2.17) and (2.16), the geostrophic region consists of a uniform westward flow. Thus, as found by Fofonoff, all eastward flow must occur in a narrow current of high relative vorticity. We restate these results in more general terms which will be useful for a comparison with the problem of the forced flow as follows: Considering only the geostrophic region, the north–south flow is completely determined ($V = 0$), but not the east–west flow ($U \sim g_1^{-1}$). If we insist that inertial boundary layers exist which will close the geostrophic flow, the (direction of the) east–west flow is determined ($g_1 > 0$). The geostrophic flow can only be westward.

A simple flow pattern obtained by Fofonoff is shown in figure 3. Note the symmetrical appearance of northward and southward flowing western and eastern coast boundary currents, as well as the necessary asymmetry of the east-

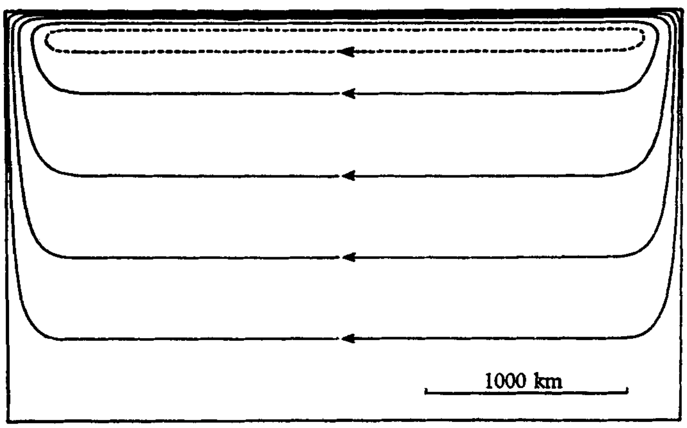

FIGURE 3. A free inertial flow pattern. Note the symmetrical character of the north–south flow in the boundary layers along the eastern and western coasts. (From Fofonoff 1954.)

ward flowing boundary current. This eastward jet has been placed on the northern boundary, i.e. the choice of $\gamma_0 = 0$ has been made. This is, of course, completely arbitrary; there is no uniqueness associated with a free solution. The eastward flow could occur on the southern boundary or in a free inertial jet at any intermediary latitude.

A useful application of the free solutions discussed here lies in the case of fluid motion which is driven by a distribution of sources and sinks imposed along its boundaries. Some special cases have been studied; in particular if a point source is placed at $\xi = 1$, $\eta = 1$ and a point sink of equal strength at $\xi = 1$, $\eta = 0$, the resulting circulation may have a boundary current along the eastern coast which is stronger than the boundary current along the western coast. The relevance of these remarks to the general ocean circulation lies in the fact that ocean basins do communicate with one another across latitudes of zero wind-stress curl, and such communication can be modelled by a source-sink distribution. This problem will be developed elsewhere, for systems driven simultaneously by wind-stress and source-sink distributions. The results are particularly relevant for southern-hemisphere oceans, e.g. they may account for the otherwise ambiguously large transport of the Benguela Current.

3. A completely inertial model

3.1. *Development for a simple wind system*

We proceed to discuss the type of forced flow allowed in an ocean basin driven by a wind-stress curl which has a maximum at some mid-latitude. The flow is assumed to have a quasi-geostrophic region and to be closed by inertial boundary layers; thus the stream-function will everywhere satisfy equation (2.10). Introducing non-dimensional variables and parameters, (2.10) takes the form

$$\epsilon[\psi_\xi(\psi_{\xi\xi}+\lambda^2\psi_{\eta\eta})_\eta-\psi_\eta(\psi_{\xi\xi}+\lambda^2\psi_{\eta\eta})_\xi]+\psi_\xi+g(\eta)=0, \tag{3.1}$$

where $\psi=(2\Omega\beta/\tau_0 a)\Psi,\quad g=(b/\tau_0)\tau',\quad \epsilon=\tau_0[ah(2\Omega\beta)^2]^{-1},$

and ξ, η, λ have been defined following equation (2.14). The transport Rossby number, ϵ, appears as the singular perturbation parameter. For the typical values $\tau_0 = 1\,\text{cm}^2\,\text{sec}^{-1}$, $a = 10^9\,\text{cm}$, $h = 10^5\,\text{cm}$, $2\Omega = 1{\cdot}4\times10^{-4}\,\text{sec}^{-1}$, $\beta = 0{\cdot}7$, we have $\epsilon = O(10^{-6})$.

The interior solution to (3.1) is obtained formally by assuming that ψ is a smooth function of (ξ,η) and thereby neglecting the ϵ-terms. We write this as

$$\psi_I(\xi,\eta)=g(\eta)[-\xi+l(\eta)], \tag{3.2}$$

where $g(\eta)\,l(\eta)$ is the non-dimensional form of $k(y)$ as discussed in § 1. The form (3.2) shows clearly that ψ^I may be zero upon one curve, $\xi = l(\eta)$, in the longitude-latitude plane, and is also convenient because we shall continue to assume that $\psi(\xi,0)=\psi(\xi,1)=0$ because $g(0)=g(1)=0$. In general, however, since ψ_I may not be made zero at $\xi = 0$, 1, we must allow for boundary layers near both the eastern and western coasts (signified by subscripts E and W respectively). Introducing the boundary-layer variables $\zeta_E=\epsilon^{-\frac{1}{2}}\xi$, $\zeta_W=\epsilon^{-\frac{1}{2}}(1-\xi)$, and recognizing that the amplitudes of ψ_E and ψ_W cannot depend upon ϵ since they must join to ψ_I as given by (3.2), the boundary-layer equation is

$$\psi_\zeta\psi_{\zeta\zeta\eta}-\psi_\eta\psi_{\zeta\zeta\zeta}+\psi_\zeta=0, \tag{3.3}$$

for either E or W subscript. Equation (3.3) is correct to $O(\epsilon^{\frac{1}{2}})$, the relative vorticity is approximated by Ψ_{xx} and the characteristic longitudinal length scale is $\epsilon^{\frac{1}{2}}a$ or tens of kilometres.

Since the boundary-layer stream functions are locally free, equation (3.3) may be integrated in the manner of § 2.6, equations (2.11), (2.12), to yield

$$\psi_{\zeta\zeta}+\eta=H(\psi). \tag{3.4}$$

In this case, however, the function $H(\psi)$ must be determined by joining to the interior solution at the boundary-layer edge in each case. The problem is complicated by the fact that the interior solution is itself not known because of the arbitrary function $l(\eta)$ appearing in (3.2). In the case of non-uniqueness, which we anticipate, the joining of the boundary layers to the interior provides relationships between the functions H_E, H_W and l which serve to restrict the interior and boundary layers allowed. To proceed simply at this point we consider only the class of interior stream functions that will be zero at some constant longitude, which may in general lie inside, outside or on a boundary of the ocean basin, i.e. we let

$$\psi_I = g(\eta)\,(-\xi + P), \tag{3.5}$$

where P is a numerical constant.

The flow patterns allowed by the coupling of inertial boundary layers with quasi-geostrophic regions will be found to be of two distinct types. These types are distinguished by whether the driving force $\tau'(y)$ varies with latitude in the same or opposite manner as the Coriolis parameter $f(y)$, i.e. depending upon the sign of $\tau''(y)/\beta$. We consider here the two simplest possibilities, by letting $\tau'(y)$ vary linearly with latitude in each case. For a single ocean basin, the two cases occur below and above the latitude of maximum wind-stress curl. Expanding $\tau'(y)$, as given by equation (2.8), about the southern and northern bounding latitudes, we obtain the relations:

near $y = 0$,

$$\tau'(y) = \frac{\tau_0}{b}\sin\frac{\pi y}{b} \doteqdot \frac{\tau_0 \pi}{b^2} y = \frac{\tau_0}{b}\pi\eta \equiv \frac{\tau_0}{b} g^+(\eta), \tag{3.6a}$$

and near $y = b$,

$$\tau'(y) \doteqdot \frac{\tau_0 \pi}{b^2}(b - y) = \frac{\tau_0}{b}\pi(1-\eta) \equiv \frac{\tau_0}{b} g^-(\eta) \tag{3.6b}$$

(see figure 1c). The superscripts $\pm$ have been introduced to distinguish between the regions where the driving force has the same or the opposite sign as β. The allowed flow patterns will now be discussed in terms of the four functions $H^{\pm}_{E,W}$ and the two interior constants $P^{\pm}$.

Case I: an equatorward half-basin

At the edge of each boundary layer, the relative vorticity becomes negligible. Setting $\psi_{\zeta\zeta} = 0$ in (3.4) and using (3.2) evaluated in terms of (3.6a), we have at the western and eastern coasts

$$\eta = H^+_W[\psi^+_I(0,\eta)] = H^+_W(\pi\eta P^+), \tag{3.7a}$$

$$\eta = H^+_E[\psi^+_I(1,\eta)] = H^+_E[\pi\eta(P^+ - 1)], \tag{3.7b}$$

whence $$H^+_W(\psi_W) = (1/\pi P^+)\,\psi_W, \quad H^+_E(\psi_E) = [1/\pi(P^+ - 1)]\,\psi_E. \tag{3.8a, b}$$

If we insert the functions (3.8a, b) into their respective boundary-layer equations obtained from (3.4), we obtain the equations

$$\psi_{\zeta\zeta} - (1/\pi P^+)\,\psi + \eta = 0, \quad \psi_{\zeta\zeta} - [1/\pi(P^+ - 1)]\,\psi + \eta = 0, \tag{3.9a, b}$$

where the subscripts E, W have been omitted on ψ, ζ of (3.9a, b) respectively. The condition that equations (3.9a, b) be of boundary-layer form, i.e. contain a real decaying exponential rather than only oscillatory homogeneous solutions is that the term in ψ alone be of opposite sign to the second derivative term. Thus, from (3.9a), $P^+ > 0$, and from (3.9b), $P^+ - 1 > 0$, will ensure that inertial boundary layers exist. It is seen that the eastern-coast condition is the strongest and contains the western-coast condition.

Case II: a poleward half-basin

We now use (3.6*b*) and proceed as above. Thus

$$\eta = H_W^-[\psi_I^-(0,\eta)] = H_W^-[\pi(1-\eta)P^-], \tag{3.10a}$$

$$\eta = H_E^-[\psi_I^-(1,\eta)] = H_E^-[\pi(1-\eta)(P^- - 1)], \tag{3.10b}$$

whence

$$H_W^-(\psi_W) = -(1/\pi P^-)\psi_W + 1, \quad H_E^-(\psi_E) = -[1/\pi(P^- - 1)]\psi_E + 1, \tag{3.11a, b}$$

and

$$\psi_{\zeta\zeta} + (1/\pi P^-)\psi + \eta - 1 = 0, \quad \psi_{\zeta\zeta} + [1/\pi(P^- - 1)]\psi + \eta - 1 = 0, \tag{3.12a, b}$$

again omitting the subscripts W, E respectively from the last two equations. For boundary-layer form, the restrictions are $P^- < 0$, $P^- - 1 < 0$. Thus in this case the western-coast condition is seen to be strongest and to contain the eastern-coast condition, in opposition to the result for case I.

Under the stated restrictions, equations (3.9*a*, *b*) and (3.12*a*, *b*) have simple exponential solutions with $\psi = 0$ at $\zeta = 0$; they join smoothly to the interior function as $\zeta \to \infty$, e.g.

$$\psi_E^+ = \pi\eta(P^+ - 1)[1 - \exp\{-[\pi(P^+ - 1)]^{-\frac{1}{2}}\zeta_E\}]. \tag{3.13}$$

Therefore the most general interior solutions of the form (3.5) are

$$\psi_I^+ = \pi\eta(-\xi + P^+) \quad (P^+ > 1); \quad \psi_I^- = \pi(1-\eta)(-\xi + P^-) \quad (P^- < 0). \tag{3.14a, b}$$

Correspondingly, the east-west component of quasi-geostrophic transport may be obtained as

$$U_I^+ = -\frac{\partial}{\partial\eta}\psi_I^+ = -\pi(-\xi + P^+) \quad (P^+ > 1), \tag{3.15a}$$

$$U_I^- = -\frac{\partial}{\partial\eta}\psi_I^- = \pi(-\xi + P^-) \quad (P^- < 0). \tag{3.15b}$$

The values of U as obtained from (3.15*a*) and (3.15*b*) are seen to be everywhere negative, as the ocean basin is contained in $0 < \xi < 1$. Thus the requirement that inertial boundary layers exist to close the quasi-geostrophic flow has served to determine the direction of the east–west flow in the quasi-geostrophic region. The flow must always be to the west (compare the discussion of § 2.5, following equation (2.17)). The wind system giving rise to the $\tau'(y)$ which we have considered is westward in the equatorward half-basin and eastward in the poleward half-basin. Thus in one case the longitudinal component of oceanic transport is in the direction of the wind and in the other case it is opposite to the direction of the wind. This is due to the interaction of the directly forced flow with the inertial boundary currents. We reiterate that a longitudinal wind determines only the latitudinal oceanic transport quasi-geostrophically.

The above results have a profound implication for the qualitative structure of allowed flow patterns over an entire ocean basin, in which there can be no net flow to the west. As will be demonstrated in § 4, curvature in the wind-stress curl will not allow an eastward flow. There must, therefore, be a breakdown of quasi-geostrophic dynamics at the latitude of the maximum of $\tau'(y)$ (where $\tau''(y)$ changes sign relative to β). The eastward flow must occur in a region of high relative vorticity, i.e. in a free inertial jet. Thus the fact that the Gulf Stream and Kuroshio leave the coast and flow eastward out to sea is simply explained by a complete inertial theory. An intense current at the latitude of maximum wind-stress curl is a required feature of all allowed flow patterns.

The simple results obtained here contain additional implications for the number and the transport of the inertial currents along the eastern and western coasts. In the equatorward half-basin there must be at least a western boundary current; the interior solution can satisfy the east-coast condition ($P^+ = 1$), but not the west-coast condition. In the poleward half-basin there must be at least an eastern boundary current; the interior solution can satisfy the west-coast condition ($P^- = 0$), but not the east-coast condition. In general, however, there will be both eastern and western boundary currents in both half basins. The direction of flow must be to the north and in the equatorward western current and the poleward-eastern current, and to the south in the equatorward-eastern and the poleward-western currents.

A final physical interpretation of the allowed $P^\pm$ values lies in the transports $T^\pm_{E,W}$ of the boundary currents. These may be determined entirely from the interior solution (3.14a, b) since ψ is zero on all coasts; hence we obtain

$$\left.\begin{aligned} T^+_W &= \psi^+_I(0,\eta) = \pi\eta P^+, & T^+_E &= \psi^+_I(1,\eta) = \pi\eta(P^+-1), \\ T^-_W &= \pi(1-\eta)P^-, & T^-_E &= \pi(1-\eta)(P^--1). \end{aligned}\right\} \tag{3.16}$$

Therefore, in the equatorward half-basin, the western current must always have greater transport than the eastern current, and in the poleward half-basin the eastern current must always have greater transport than the western current. What happens is that in each half-basin the boundary current which must always be present transports the amount of fluid directly forced by the wind, as well as sharing in a recirculation phenomenon involving also the mid-latitude current, the other boundary current, and a broad flow across the major part of the ocean. It should be noted that the transport discrepancy for the Gulf Stream and Kuroshio which one obtains using the frictional theory discussed in § 2.3 is not contained in the inertial theory. The Sverdrup–Munk transport function corresponds to the case $P^+ = 1$. As this is the minimum allowed value of P^+ it yields the minimum allowed value of T^+_W, which may be arbitrarily larger.

The lack of uniqueness inherent in this discussion is a real feature of a complete steady-state inertial theory. The ambiguity will, however, be partially removed by the inclusion of additional physics when the effects of friction are considered. It will be shown that an eastern boundary current in the equatorward half-basin is required and correspondingly that the minimum allowed value for T^+_W is greater than the Sverdrup–Munk transport.

In figure 4, a sketch is presented of the simplest stream-line pattern allowed by the inertial theory, for $P^+ = 1$, $P^- = 0$. Note that (for a northern-hemisphere ocean) the mid-latitude jet is fed from the north and discharges to the south. Note, however, that there is no transport across the zero stream-line which forms the axis of the jet, i.e. on a completely inertial model, the half-basins do not communicate. The striking difference between this pattern and Sverdrup–Munk transport streamlines may be seen from figure 2*b*. Since, in general, boundary currents will be present also on the north-western and south-eastern coasts, a schematic comparison with figure 2*a* shows that the Kuroshio, the Kuroshio extension or North Pacific current, the California current, the Alaska current and the Oyashio may be simply explained on the inertial theory.

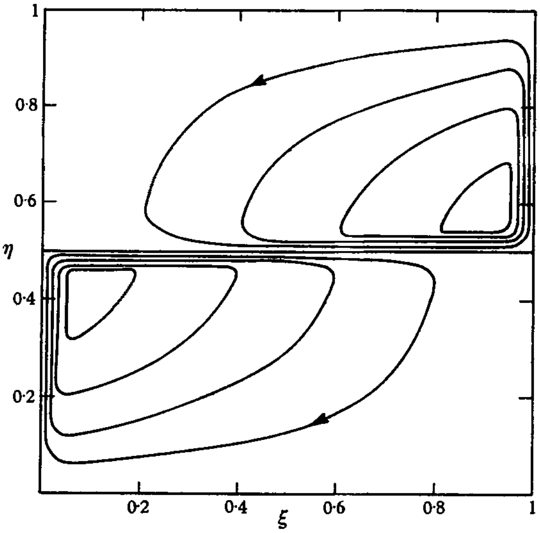

FIGURE 4. Transport streamlines in the (ξ, η) square ocean as given by the inertial theory. The double gyre with a mid-latitude inertial jet is the simplest response to the trigonometric wind-stress curl of figure 1*b*.

3.2. *The mid-latitude jet*

Since the broad flow over the open ocean transports mass only to the west, the transport to the east must occur in a region of high relative vorticity. This inertial current must occur at about the latitude of maximum wind-stress curl, and will be 'free' in the sense that no physical boundary or local boundary condition is directly forcing the singular region to occur there. In the sense that there must occur an influx and efflux of mass along the sides of the jet, it will not be free. To explore the dynamics and scale of such a jet we shall assume in this section that the direction of the strong current is purely longitudinal, i.e. that large gradients occur only in the latitudinal direction.

An unusual feature of the dynamics of this current system is that the variation of the Coriolis parameter is of primary importance, even though the phenomenon occurs within a latitude belt less than 100 km in width. This is because the importance of the β-effect is not measured directly by the percentage variation of f over the latitudinal extent of the current, but is measured rather by the contribution to the overall vorticity balance made by the planetary vorticity tendency, $\beta\Psi_x$, in the region. Over the major part of the ocean, both north and south of the narrow current, the advection of relative vorticity is completely negligible with respect to the planetary vorticity tendency. The width of the current is a natural scale whereby the advection becomes comparable. This is similar in principle to the vorticity balance in a coastal inertial current, but differs in that in a coastal current $\beta\Psi_x$ becomes larger than in the open ocean, while in the mid-latitude current it does not. Correspondingly, the relative vorticity (as a point function of latitude and longitude) is larger in a coastal jet, but since the mid-latitude jet is broader, the total vorticity transport by both types of currents will be the same order of magnitude (with respect to dependence upon the Rossby number, ϵ).

To present a formal description, we introduce a scaled latitudinal variable, and obtain the approximate form of equation (3.1), which becomes

$$\lambda^2(\psi^J_\xi \psi^J_{XXX} - \psi^J_X \psi^J_{\xi XX}) + \psi^J_\xi + g(\eta_0) = 0, \tag{3.17}$$

where $X = \epsilon^{-\frac{1}{3}}(\eta - \eta_0)$, and the superscript J refers to the region of the mid-latitude jet. The amplitude of ψ must again be independent of ϵ in order to join to the solutions on either side of the jet. The equation is correct to $O(\epsilon^{\frac{2}{3}})$. The dependence upon λ can, of course, be removed by a scaling transformation to a latitudinal variable $\lambda^{-\frac{2}{3}}X$.

It is of some interest to demonstrate that equation (3.17) may be reduced to an ordinary differential equation by means of a similarity transformation. Since such a transformation depends upon the interior function of the transverse variable at the boundary-layer edge, and since this function must differ on the poleward and equatorward sides of the jet, it is necessary to introduce separate variables to measure distance away from the central zero-streamline in each direction. We illustrate by the equatorward directed similarity transformation. Using an interior solution ψ^+_I of the form (3.5) and the notation

$$g(\eta_0) = g_0, \quad s = X[\lambda^2(\xi - P^+)]^{-\frac{1}{3}}, \quad \psi^J = (\xi - P^+)[F(s) - g_0], \tag{3.18}$$

(3.17) transforms to

$$g_0 F'''' - F'''F + \tfrac{1}{3}F'F'' + \tfrac{1}{3}sF' + F = 0, \tag{3.19}$$

where a prime denotes total differentiation with respect to s. The boundary conditions are $F(\infty) = 0$ and continuity of solution at $s = 0$. It may be shown from a consideration of the asymptotic form of (3.19) that an exponentially decaying solution does exist, but this will be omitted here since a more complete mathematical treatment of the inertial flow is to be presented in §4 (see equation (4.16)). The relevance of the similarity transformation to the general argument is that the asymptotic approach to the interior solution will be along lines of constant s rather than constant X. From (3.18) and an analogous poleward-directed transformation it may be seen how the mid-latitude jet will broaden to the east as it loses mass on the equatorward side, and how it will narrow to the east on the poleward side as it gains mass. This structure has been incorporated into the sketch presented in figure 4.

3.3. *The integrated vorticity constraint and the necessity for diffusive transfer*

We have up to this point discussed features of the general ocean circulation for which the model described by equation (3.1) is appropriate, i.e. the nature of the flow in the quasi-geostrophic region covering the major oceanic areas and the position and relative strength of the intense streams of high relative vorticity necessary to complete the flow field. There are, however, features of the circulation which cannot be deduced by such a simple model. For example, the values of $P^{\pm}$, or the amount of recirculation in each half-basin, are not determinate. Furthermore, the velocity distribution of the inertial solution discussed contains features which cannot occur in any real fluid; in each inertial boundary current, the velocity tangential to the coast reaches a maximum at the coast, and in the mid-latitude jet there is a discontinuity of tangential velocity across the zero-streamline. Any viscosity, no matter how small, must certainly smooth these discontinuities. If the (turbulent) viscosity is so small that this is the only role it plays in the general circulation, then viscous phenomena are essentially uninteresting from the large-scale point of view. We shall see, however, that this is not the case, by considering the constraint imposed upon the fluid motion by integrating the vorticity equation over the entire area of the ocean basin.

To perform the integration we note that the inertial terms may be rewritten, with the notation

$$\Delta = \partial^2/\partial\xi^2 + \lambda^2\partial^2/\partial\eta^2,$$

as

$$\psi_\xi\Delta\psi_\eta - \psi_\eta\Delta\psi_\xi = \partial(\psi\Delta\psi_\eta)/\partial\xi - \partial(\psi\Delta\psi_\xi)/\partial\eta; \tag{3.20}$$

we then multiply (3.1) by $d\xi d\eta$ and integrate over the range $(0, 1)$ in both variables to give

$$\epsilon\int_0^1 d\eta[\psi(\Delta\psi_\eta)_\eta]_{\xi=0}^{1} - \epsilon\int_0^1 d\xi[\psi(\Delta\psi_\xi)]_{\eta=0}^{1} + \int_0^1 d\eta\,\psi_{\xi=0}^{1} + \int_0^1\int_0^1 d\xi d\eta\, g(\eta) = 0 \tag{3.21}$$

or, since $\psi = 0$ on all boundaries, all terms vanish but the last, and (3.21) reduces to $\int_0^1 d\eta\, g(\eta) = 0$. But this is clearly impossible as $g(\eta)$ is positive definite, the wind-stress curl being everywhere of one sign over the ocean basin. Thus a completely inertial model violates the integrated vorticity constraint. The integration (3.21) is an explicit statement, in terms of the specific inertial model adopted here, of general objections which have been raised against complete inertial models on the grounds that integral conservation laws must be violated. It should be noted that (3.21) is an integral vorticity statement, and not an integral angular momentum statement, which need not be violated inertially (Morgan 1956, § 2).

The fact that the inertial model cannot satisfy the integrated constraint does not, of course, mean that the model is inadequate to describe those aspects of the flow to which it is applicable. Equation (3.1) must be regarded as an approximate form of the more complete equation (2.4). It provides for a local balance between a distributed source of vorticity (the integrated wind-stress curl) and the divergence of an advective flux of absolute vorticity, $\nabla^2\Psi + f$. However, the distributed vorticity source is everywhere of one sign and correspondingly there is a net input of vorticity into the fluid by the wind-stress. The vorticity advection, although balancing the source locally, can only redistribute vorticity internally, it cannot absorb vorticity. The only possible sink of vorticity is a diffusion into the boundaries.† The diffusive transfer process must, therefore, be included if the integrated balance is to be considered.

The inertial theory must therefore be regarded only as the inviscid core of a velocity field which also contains thin frictional layers, and when thus completed, will contain no physical paradoxes. In the limit of very small (horizontal) friction, which appears empirically to be the correct limit for the general ocean circulation, the flow field outside the frictional regions is determined essentially independently of friction. Mathematically, there is a three-scale problem, and the narrowest boundary-layer contribution (frictional scale) can fit on to whatever is required by the mutually-determined solutions from the two broader scales (quasi-geostrophic or geometric and inertial). Physically, however, the effects of friction on the general circulation is of great interest, since the ocean is a turbulent system which generates its own 'friction' and in which there is present only the minimum amount of turbulent diffusion necessary to satisfy overall conservation laws. In a sense the mean flow has control over the turbulence, rather than the turbulence controlling the mean flow. This is true in the region of the mid-latitude jet. In the region of the coasts the nature of the diffusive boundary layer is such that it contributes to the overall vorticity balance in a manner independent of the amount of friction present. To illustrate these ideas, we shall at this point introduce a process of diffusion by means of a constant coefficient of kinematic eddy viscosity, ν. We shall allow the eddy viscosity to have different constant values in the different regions where diffusion contributes to the local vorticity balance. In no sense do we imply that this approaches an accurate description of the turbulence. It does, however, provide a semi-quantitative description of the relation of diffusive transport to the other mechanisms of vorticity transfer which are present.

Thus we add to the vorticity equation a term $-\nu\Delta^4\Psi$, or in non-dimensional form, we replace equation (3.1) by

$$-\gamma\Delta\Delta\psi+\epsilon(\psi_\xi\Delta\psi_\eta-\psi_\eta\Delta\psi_\xi)+\psi_\xi+g(\eta)=0, \tag{3.22}$$

where $\gamma=\nu b/2\Omega\beta a^3$ is obviously related to a transport Ekman or Taylor number. We may immediately obtain relationships between γ and ϵ for the inertial currents discussed in §§3.1 and 3.2 to be a valid approximation to the flow, i.e. obtain bounds on the eddy viscosity in coastal and mid-latitude regions. Near a coast, $\partial/\partial\xi=O(\epsilon^{-\frac{1}{2}})$, and in the free jet, $\partial/\partial\eta=O(\epsilon^{-\frac{1}{3}})$. The sizes of the largest diffusive terms relative to the important advective terms are $\gamma\epsilon^{-\frac{3}{2}}$ and $\gamma\epsilon^{-\frac{4}{3}}$ respectively. Thus,

in a meridional inertial current:

$$\gamma<\epsilon^{\frac{3}{2}}\quad\text{or}\quad\nu<(\tau_0ah^{-1})^{-\frac{3}{2}}b^{-1}(2\Omega\beta)^{-2}, \tag{3.23a}$$

in a latitudinal inertial current:

$$\gamma<\epsilon^{\frac{4}{3}}\lambda^{-2}\quad\text{or}\quad\nu<(\tau_0h^{-1})^{\frac{4}{3}}a^{\frac{11}{3}}b^{-3}(2\Omega\beta)^{-\frac{5}{3}}. \tag{3.23b}$$

† We shall consider below only the possibility of diffusion into lateral boundaries, although it is conceivable that some vorticity may be diffused into the ocean bottom via barotropic eddies.

The simplest way to illustrate the nature of coastal frictional regions is to regard them from the point of view of boundary layers on the inertial current. Consequently, we introduce the inertial-layer variable, ζ_E or ξ_W, and add the diffusive term to equation (3.3), retaining only the term $\partial^4/\partial\xi^4$ in Δ. Thus

$$-\Gamma\psi_{\zeta\zeta\zeta\zeta}+\psi_\zeta\psi_{\zeta\zeta\eta}-\psi_\eta\psi_{\zeta\zeta\zeta}+\psi_\zeta = 0, \tag{3.24}$$

where $\Gamma = \gamma\epsilon^{-\frac{3}{2}}$ measures the effect of friction, and may be seen to be simply an inverse Reynolds number based on the eddy viscosity, the transport velocity at the quasi-geostrophic edge of the inertial layer, and the width of the inertial layer as a length scale. For large Reynolds number, frictional effects will be confined near the boundary $\zeta = 0$ and we introduce a scaled variable $\mu = \Gamma^r\zeta$ for this region.† The amplitude of the stream function must also be scaled in Γ; let $\psi = \Gamma^s\chi(\mu, \eta)$, where χ is a smooth function. The additional requirement is that the amplitude of the velocity parallel to the coast be the same in the frictional layer, where it is to be brought down to zero, as it is in the inertial layer, i.e. $\psi\zeta = \Gamma^{s+r}\chi\mu$ must remain independent of Γ, or $s = -r$. We then find that

$$\mu = \Gamma^{-\frac{1}{2}}\zeta, \quad \chi = \Gamma^{\frac{1}{2}}\psi, \tag{3.25}$$

and the approximate equation becomes

$$-\chi_{\mu\mu\mu\mu}+\chi_\mu\chi_{\mu\mu\eta}-\chi_\eta\chi_{\mu\mu\mu} = 0, \tag{3.26}$$

the equation for an ordinary non-rotating fluid boundary layer in which viscosity and inertia are balanced. This is seen to occur when the velocity normal to the coast, which is $O(1)$ in the quasi-geostrophic interior, is $O(\Gamma^{\frac{1}{2}})$.

For the integrated vorticity balance, we must replace (3.21) by the integral of (3.22) over the ocean basin. To evaluate the term $\Delta\Delta\psi$ we note that the contribution will be negligible except in coastal frictional boundary layers where the full term may be replaced approximately by $\psi_{\xi\xi\xi\xi}$. Thus the balance

$$\gamma\int_0^1 [\psi_{\xi\xi\xi}(0,\eta)-\psi_{\xi\xi\xi}(1,\eta)]\,d\eta+\int_0^1 g(\eta)\,d\eta = 0 \tag{3.27}$$

obtains. We note first that (3.26) is entirely independent of the eddy viscosity. This occurs because at the boundaries, from (3.25), we have that $\psi \sim \nu^{\frac{1}{2}}$, $\partial/\partial\xi \sim \nu^{-\frac{1}{2}}$. Since $\gamma \sim \nu$, the combination $\gamma\psi_{\xi\xi\xi}$ does not contain ν. As the eddy viscosity becomes smaller, the gradients sharpen in the viscous layer in such a manner as always to maintain the integrated vorticity constraint. Furthermore, we can obtain the value of $\psi_{\xi\xi\xi}$ on the coasts, explicitly in terms of the quasi-geostrophic interior solutions, by integrating once the full boundary-layer equation (3.24). The integral may be performed simply by rewriting the inertial terms as

$$\psi_\zeta\psi_{\zeta\zeta\eta}-\psi_\eta\psi_{\zeta\zeta\zeta} = \partial(\psi_\zeta\psi_{\zeta\eta}-\psi_\eta\psi_{\zeta\zeta})/\partial\zeta.$$

Since on the coasts in the presence of friction $\psi_\zeta = \psi_\eta = 0$, and interior to the inertial currents the relative vorticity is negligible, these terms yield nothing when integrated across the frictional and inertial boundary layers, from the coast to the quasi-geostrophic interior. From (3.5), (3.24) and (3.25),

$$\gamma\psi_{\xi\xi\xi}(0,\eta) = -g(\eta)P, \quad \gamma\psi_{\xi\xi\xi}(1,\eta) = -g(\eta)(P-1), \tag{3.28a, b}$$

and in virtue of the restrictions $P^+ > 1$, $P^- < 0$, it is seen from (3.27) that the equatorward-western and poleward-eastern boundaries diffuse into the ocean

† See note added in proof stage at end of paper (p. 80).

vorticity of opposite sign to that of the distributed wind source, while the equatorward-eastern and poleward-western boundaries diffuse vorticity of the same sign as the wind.

Equations (3.28*a*, *b*) provide expressions for the diffusive contributions to (3.27) valid everywhere on the western and eastern coasts except for a narrow latitude belt of width $\epsilon^{\frac{1}{4}}$ situated about the centre of the mid-latitude jet, or at the latitude of maximum wind-stress curl. If the contributions from these short stretches of the coasts be neglected, which seems *a priori* reasonable, (3.27) may be evaluated. Under the assumption of a wind-stress curl which is symmetric about the middle latitude $\eta = \frac{1}{2}$, e.g. (2.8), the vorticity integral (3.27) becomes simply

$$\tfrac{1}{2}\{-P^+ - P^- + (P^+ - 1) + (P^- - 1)\} + 1 = 0, \tag{3.29}$$

which is seen to be automatically satisfied for any amounts of recirculation in the two half-basins. There is, however, another point of view which is physically more plausible than that adopted here for the evaluation of (3.27), and which yields a minimum lower bound on P^+ which is higher than that obtained from inertial considerations alone. The detailed argument will be reserved for § 4 and only an outline presented here. The physical basis is the diffusion of vorticity in the region of the mid-latitude jet.

As has been mentioned above, diffusion must certainly act to eliminate the tangential velocity discontinuity at the mid-latitude zero-streamlines. That diffusion do no more than this at mid-latitudes is the basis for the inequality (3.23*b*), under which a very narrow frictional region will be sandwiched between two inertial streams. That this cannot be the case can be seen by considering integrated vorticity constraints for partial areas of the ocean basin, e.g. in the lower half-basin, over the three frictional sub-layers (lying along $\xi = 0,\ 1,\ \eta = \frac{1}{2}$) on the one hand and over the quasi-geostrophic pulse inertial current regions on the other hand. Each such region will be found to be separately out of balance, the vorticity diffused out of the ocean at the boundaries not having been allowed to diffuse out of the body of the fluid into which it was put by the distributed source.† It is plausible to assume that this occurs in the region of the mid-latitude jet, since boundary currents are known to be essentially inertial. If this be the case, the inequality (3.21*b*) must at least become an equality; hence $\nu = 10^7\,\mathrm{cm^2/sec}$ (evaluated for numbers typical of the North Atlantic). Thus the theory of the mean flow yields a crude but quantitative prediction of the amount of turbulence which must be present. A simple dimensional argument from the largest-scale meanders of the Gulf Stream after it leaves the coast at Cape Hatteras gives $\nu = 10^8\,\mathrm{cm^2/sec}$ as a crude empirical estimate of the eddy viscosity in this region.

Furthermore, under the assumption that diffusion does occur in the region of the mid-latitude current in a wide enough region to destroy the purely inertial character of the jet, it is highly unlikely that only a minimum of vorticity will be diffused. Consider therefore the possibility that the vorticity advected into

† From a formal point of view we may say that every streamline of the flow must go through a frictional region, since inertial currents conserve (potential) vorticity along streamlines.

the narrow core of the jet from the western walls has essentially all diffused and broadened the jet as it reaches the eastern wall. Assuming the separation of the eastern coastal currents into inertial and frictional regions to be maintained, it may be inferred that there can be no net diffusion of vorticity into the ocean along the eastern coast. In such a case the vorticity diffusion along the short stretch of eastern coast around the mid-latitude is not negligible; in fact, it cancels the contribution (3.28). As will be argued in §4, from the nature of better understood but similar stagnation points of fluid, flow, it is possible for this to occur while the corresponding contribution from the western coast does remain negligible. In this extreme case, we replace (3.29) by a relationship containing no eastern coast contribution,

$$\tfrac{1}{2}(-P^{+}-P^{-})+1 = 0, \quad \text{or} \quad P^{+} = 2-P^{-}, \tag{3.30}$$

and, since $P^{-} \leqslant 0$, the minimum value of $P^{+} = 2$. In this case the transport of the equatorward-western current (Gulf Stream or Kuroshio) is twice that given by the purely frictional theory, and all the vorticity input by the wind is balanced by diffusion in this region. In a less extreme case, the vorticity balance in the eastern coastal currents could be partially maintained by a breakdown of the strict separation into purely inertial and frictional streams. The transports of the Gulf Stream and Kuroshio would then be more than, but less than twice, that given by the Sverdrup–Munk value. Furthermore, this would yield eastern currents which are broader and less well defined than western currents, which is apparently the case observationally. This view may be supported by the fact that the eastern currents have a downstream mass efflux, and would tend to be more unstable than their western counterparts.

An interesting consequence of this discussion is that in each half-basin a balance does not obtain between wind-stress curl and coastal diffusion of vorticity. A balance is maintained only by a meridional transport of vorticity by the meanders or eddies in the mid-latitude jet. If it can be shown that these meanders do indeed effect a transfer of vorticity between low and high latitudes, then they play, in the general circulation of the ocean, the same vital role that large scale meanders in the jet stream are known to play in the general circulation of the atmosphere.

In the following sections we shall develop the inertial flows with additional generality and then present further arguments as to the implication of turbulent diffusion.

4. The inertial boundary-layer theory

We return now to discuss, for a more general geometry and wind-stress distribution, the model described in §2.

The differential equation (3.1) for the transport stream function ψ is

$$\psi_{\xi}(\epsilon\Delta\psi+f)_{\eta}-\psi_{\eta}(\epsilon\Delta\psi+f)_{\xi}+g(\eta) = 0, \tag{4.1}$$

where $f(\eta)$ is a redefined non-dimensional form of the latitudinally varying vertical component of the earth's rotation. The boundaries lie at $\xi = \xi_1(\eta)$, $\xi = \xi_2(\eta)$, $\eta = 0$ and $\eta = L$. The former of these are continental barriers on which ψ must vanish and the latter are those latitudes at which $g(\eta)$, the curl of the wind stress, vanishes. We shall see that ψ may also be taken to vanish at these latitudes. The number L determines the length scaling (the quantity b of equation (3.1) times L is the distance from $\eta = 0$ to $\eta = L$), and we use the same scale in the longitudinal direction so that $a = b$ and $\lambda = 1$ (a, b, λ are quantities introduced with equation (3.1)).

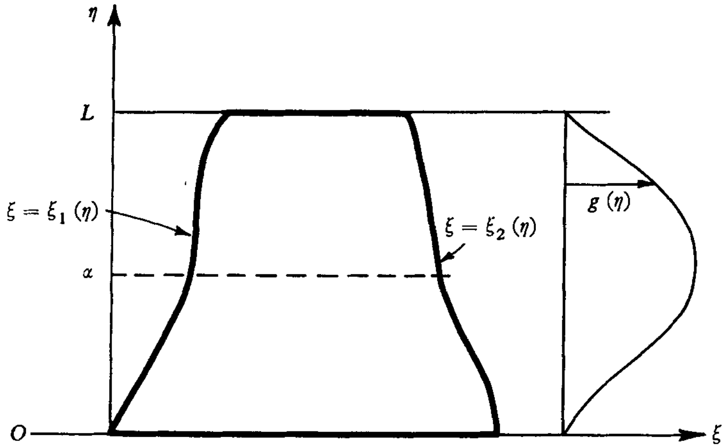

FIGURE 5. Geometry of ocean basin and wind-stress curl $g(\eta)$.

Both because of the mathematical structure of the problem [i.e. because the most highly differentiated terms in (4.1) have a very small coefficient (Carrier 1953)] and because of the foregoing discussion, we anticipate that the circulation pattern will display a boundary-layer structure and that ψ can be written in the form

$$\psi(\xi, \eta) = \psi^{(0)}(\xi, \eta) + \psi^{(1)}(\zeta_1, \eta) + \psi^{(2)}(\zeta_2, \eta), \tag{4.2}$$

where $\zeta_j = [\xi - \xi_j(\eta)]\,\epsilon^{-\frac{1}{2}}$. In choosing this representation for ψ, we deliberately do *not* allow sufficient generality because we wish to *deduce* the necessity of a more highly subdivided structure. We shall find, in fact, that the interior domain must be divided into two regions (as in §3) and that inertial currents (boundary layers) will occupy not only the eastern and western boundaries, but that an intense current will also flow at constant latitude across the ocean basin at $\eta = \alpha$, the latitude of maximum wind-stress curl.

Since $\psi^{(1)}$ and $\psi^{(2)}$ describe boundary layers, they must die out exponentially as $\zeta_1 \to \infty$, $\zeta_2 \to -\infty$, respectively.

We substitute (4.2) into (4.1) and segregate into three groups the various contributions which emerge, to give

$$\begin{aligned}&[\psi^{(0)}_{\xi} f'(\eta) + g(\eta) + O(\epsilon^{\frac{1}{2}})]\\ &\quad + \epsilon^{-\frac{1}{2}}[(\psi^{(1)}_{\zeta_1}\psi^{(1)}_{\zeta_1\zeta_1\eta} - \psi^{(1)}_{\eta}\psi^{(1)}_{\zeta_1\zeta_1\zeta_1})\,q_1(\eta) - \psi^{(0)}_{s1}(\eta)\,\psi^{(1)}_{\zeta_1\zeta_1\zeta_1} + \psi^{(1)}_{\zeta_1} f'(\eta) + O(\epsilon^{\frac{1}{2}})]\\ &\quad + \epsilon^{-\frac{1}{2}}[(\psi^{(2)}_{\zeta_2}\psi^{(2)}_{\zeta_2\zeta_2\eta} - \psi^{(2)}_{\eta}\psi^{(2)}_{\zeta_2\zeta_2\zeta_2})\,q_2(\eta) - \psi^{(0)}_{s2}\,\psi^{(2)}_{\zeta_2\zeta_2\zeta_2} + \psi^{(2)}_{\zeta_2} f'(\eta) + O(\epsilon^{\frac{1}{2}})] = 0. \end{aligned} \tag{4.3}$$

The first bracket contains all terms which are not exponentially small far from the lateral boundaries; the second contains those terms which are not small near the western boundary; the third contains the terms which determine the structure of the eastern boundary layer; the quantities q_j and ψ_{sj} appearing in (4.3) are $1+[\xi_j'(\eta)]^2$ and the directional derivative of $\psi^{(0)}$ along the boundary $\xi = \xi_j(\eta)$, respectively. Equation (4.1) and this discussion imply that

$$\psi_\xi^{(0)} f'(\eta) = g(\eta) = 0, \tag{4.4}$$

and, hence, that
$$\psi^{(0)} = g(\eta)\,[a(\eta)-\xi]/f'(\eta), \tag{4.5}$$

where $a(\eta)$ has yet to be determined.†

The second bracket of (4.3) must also vanish and so

$$(\psi_{\zeta_1}^{(1)}\psi_{\zeta_1\zeta_1\eta}^{(1)} - \psi_\eta^{(1)}\psi_{\zeta_1\zeta_1\zeta_1}^{(1)})\,q_1(\eta) - \psi_{s1}^{(0)}(\eta)\,\psi_{\zeta_1\zeta_1\zeta_1}^{(1)} + \psi_{\zeta_1}^{(1)} f'(\eta) = 0. \tag{4.6}$$

The treatment of this equation is simplified by writing

$$\psi_{\zeta_1\zeta_1}^{(1)} = F[\psi^{(1)}, \eta], \tag{4.7}$$

whereupon (4.6) becomes

$$\psi_{\zeta_1}^{(1)}\,[q_1 F_\eta - \psi_{s1}^{(0)}(\eta)\,F_{\psi^{(1)}} + f'(\eta)] = 0. \tag{4.8}$$

It follows that
$$F(\psi^{(1)}, \eta) = -W_1(\eta) + G_1[\psi^{(1)} + \chi_1(\eta)], \tag{4.9}$$

where
$$W_1(\eta) = \int_0^\eta \frac{f'(\eta)}{q_1(\eta)}\,d\eta, \quad \chi_1(\eta) = \int_0^\eta \frac{\psi_{s1}^{(0)}(\eta)}{q_1(\eta)}\,d\eta,$$

and (thus far) $G_1(u)$ is any differentiable function of u. However, if $\psi^{(1)}$, the solution of (4.7), is to die out as $\zeta_1 \to \infty$, $F(0, \eta)$ must be zero. Thus

$$G_1[\chi_1(\eta)] = W_1(\eta). \tag{4.10}$$

If we define $\eta_1^*(\chi_1)$ to be the inverse of $\chi_1(\eta)$, then

$$G_1(u) = W_1[\eta_1^*(u)]. \tag{4.11}$$

In order to see simply the implications of (4.11) which are vital to the determination of the current structure, it is convenient to consider a simple but realistic example. Let $L = \pi$, $\xi_1(\eta) = 0$, $\xi_2(\eta) = \text{const.} = \xi_2$, $g(\eta) = \sin\eta$, $f = f_0 + \eta$. This $g(\eta)$ is a reasonable qualitative approximation to the observed wind-stress curl over the major ocean basins, and the slowly varying trigometric form of the correct $f(\eta)$ is approximated by η, as is customary on a β-plane model. With these definitions,
$$W_1(\eta) = \eta, \quad \chi_1(\eta) = \psi_\eta^{(0)}(0, \eta) = A_1 \sin\eta,$$

and we arbitrarily take $a(\eta)$ to be the constant $A_1 > 0$; we defer until later a more comprehensive discussion of this choice. Thus $G_1(u) = \sin^{-1}(u/A_1)$, so that

$$F(\psi^{(1)}, \eta) = -\eta + \sin^{-1}\{A_1^{-1}[\psi^{(1)}(\zeta_1, \eta) + \psi^{(0)}(\eta, 0)]\}, \tag{4.12}$$

and
$$\psi_{\zeta_1\zeta_1}^{(1)} = -\eta + \sin^{-1}\{A_1^{-1}[\psi^{(1)}(\zeta_1, \eta) + \psi^{(0)}(\eta, 0)]\}. \tag{4.13}$$

Equation (4.13) can be integrated explicitly by quadratures. In particular, multiplication by ψ_{ζ_1} and integration over ζ_1 yields

$$[\psi_{\zeta_1}^{(1)}(\zeta, \eta)]^2 - [\psi_{\zeta_1}^{(1)}(0, \eta)]^2 + 2\eta[\psi^{(1)}(\zeta_1, \eta) - \psi^{(1)}(0, \eta)] = 2A[u\sin^{-1}u + (1-u^2)^{\frac{1}{2}} - 1],$$

where $u = \sin\eta + A_1^{-1}\psi^{(1)}(\zeta, \eta)$.

† Note that the choice of $\psi^{(0)}$ in this form, with $a(\eta)$ bounded in $0 \leqslant \eta \leqslant 1$, implies that $\psi^{(0)}(\xi, 0) = \psi^{(0)}(\xi, L) = 0$. Thus $\eta = 0$ and $\eta = L$ are streamlines of flow. We limit ourselves, in this investigation, to such $\psi^{(0)}$ but we note in doing so that no other choice of $\psi^{(0)}$ permits the elimination of any of the inertial currents which are obtained in the following.

Since

$$\psi^{(1)}(\infty,\eta) = 0 \quad \text{and} \quad \psi^{(1)}(0,\eta) = -\psi^{(0)}(0,\eta), \quad [\psi^{(1)}_{\zeta_1}(0,\eta)]^2 = 2A_1(1-\cos\eta).$$

Thus, at $\eta = \frac{1}{2}\pi$, the velocity at the continental barrier is proportional to $A_1^{\frac{1}{2}}$.

Using (4.13), when $\eta < \frac{1}{2}\pi$ and we define $\sin^{-1}(0) = 0$, $\psi^{(1)}_{\zeta_1\zeta_1}(0,\eta) < 0$, and, as $\zeta_1 \to \infty$, $\psi^{(1)}_{\zeta_1\zeta_1} \to 0$. However, when $\eta > \frac{1}{2}\pi$, $\sin^{-1}\{A_1^{-1}[\psi^{(1)}+\psi^{(0)}]\}$ begins with the value zero at $\zeta_1 = 0$ and tends to $\pi-\eta$ as $\psi^{(1)} \to 0$, thus making it impossible for $\psi^{(1)}_{\zeta_1\zeta_1}$ to vanish. In other words, equation (4.13) has no solution which approaches zero as $\zeta_1 \to \infty$. *Thus the inertial boundary layer on the western side of the ocean can continue poleward only up to the latitude where $g(\eta)$ has its maximum value.* For more general $f(\eta)$, $g(\eta)$, $\xi_1(\eta)$, the greatest poleward penetration of the western current cannot exceed $\eta = \alpha$, the latitude at which $[A(\eta)-\xi_1(\eta)]\,g(\eta)/f'(\eta)$ has its maximum value.

The equations which govern the behaviour of the eastern current are identical with those for $\psi^{(1)}$ except that the index 2 replaces the index 1. The solution $\psi^{(2)}$ of that equation will die out as $\zeta_2 \to -\infty$ provided $a(\eta)-\xi_2(\eta) > 0$; $\psi^{(2)}$ can be found by integrating (4.13).

Once again, by identical arguments, the current (which flows towards the equator) can be continued only as far poleward as that latitude where $[a(\eta)-\xi_2(\eta)]\,g(\eta)/f'(\eta)$ has its maximum. Mass conservation in the large can now be included in the theory only if a current flows eastward from the poleward terminus of the western current to that of the eastern current. This current can be sought as still another boundary layer using the description

$$\psi = \psi^{(0)}+\psi^{(1)}+\psi^{(2)}+\psi^{(3)}(\xi,\sigma),$$

where $\sigma = \epsilon^{-\frac{1}{3}}[\eta-\alpha]$.

With this modification of ψ, equation (4.3) will be modified by the addition of another bracket, which must vanish independently of the other brackets of that equation. Thus

$$\psi^{(3)}_{\xi}\psi^{(3)}_{\eta\eta\eta} - \psi^{(3)}_{\eta}\psi^{(3)}_{\eta\eta\xi} + \psi^{(0)}_{\xi}(\xi,\alpha)\,\psi^{(3)}_{\eta\eta\eta} + f'(\alpha) = 0. \tag{4.14}$$

This equation admits a similarity solution of the form

$$\psi^{(3)} = \{a(\alpha)-\xi\}\,h[\sigma\{a(\alpha)-\xi\}^{-\frac{1}{3}}] = [a(\alpha)-\xi]\,h(\tau). \tag{4.15}$$

Substitution of this representation of $\psi^{(3)}$ into (4.14) yields the equation

$$(1+h)\,h''' - \tfrac{1}{3}h'h'' + h - \tfrac{1}{3}\tau h' + O(\epsilon^{\frac{1}{3}}) = 0.$$

The relevant solution of this equation is one whose asymptotic behaviour for large negative τ (as governed by the linear terms of (4.15)) is

$$h_1 \sim C\tau^{-\frac{1}{4}}\exp\,(2\tau^{\frac{3}{2}}/3\surd 3), \tag{4.16}$$

and for which $h_1(0) = g(\alpha)/f'(\alpha)$. The details of h_1 can be found by numerical integration but, once again, such details provide no particular advantage.

The description of the flow for $\eta < \alpha$ which is provided by $\psi_0+\psi_1+\psi_2+\psi_3$ is unsatisfactory only in the poleward corners where the boundary layers join. The efflux of vorticity from the western current defined by $\psi^{(1)}(\zeta_1, \alpha)$ differs markedly from that entering the horizontal current described by $\psi^{(3)}(0, \sigma)$ and a similar situation prevails near the point $[\xi_2(\alpha), \alpha]$. The former is of order unity and the latter of order $\epsilon^{\frac{1}{3}}$. Before resolving this difficulty by invoking frictional considerations, we must discuss the flow above $\eta = \alpha$.

The analysis proceeds precisely as before; we denote the stream function by ϕ instead of ψ and use $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, respectively, to describe the interior flow and the western, eastern and central inertial currents. The independent variables ζ_1, ζ_2, σ are defined precisely as before; in fact,

$$\phi^{(0)}(\xi,\eta) = [a^*(\eta)-\xi]\,g(\eta)/f'(\eta). \tag{4.17}$$

$\phi^{(1)}(\xi,\eta)$ is the solution of $\phi^{(1)}_{\zeta_1\zeta_1} = F(\phi^{(1)},\eta)$ and

$$q_1 F_\eta - \phi^{(0)}_{s1}(\eta)\,F_{\phi^{(1)}} + f'(\eta) = 0. \tag{4.18}$$

It follows that

$$F(\phi^{(1)},\eta) = -W(\eta) + G[\phi^{(1)} + \chi_1(\eta)], \tag{4.19}$$

where

$$W = \int_0^\eta \frac{f'(\eta)}{q_1(\eta)}\,d\eta, \quad \chi_1(\eta) = -\int_0^\eta \frac{\phi^{(0)}_{s1}(\eta)}{q_1(\eta)}\,d\eta,$$

and that $G(u)$ is defined in terms of η^{**} the inverse of $\psi^{(0)}_{s1}(\eta)$ by the relation

$$G(u) = W[\eta^{**}(u)]. \tag{4.20}$$

For the special case treated before, $W(\eta) = \eta$ and $G(u) = \sin^{-1}(u/B_1)$, where we now take $a^*(\eta) = B_1$ in the description of $\phi^{(0)}$, and we must now define $\sin^{-1}(0) = \pi$. However, with these definitions of W and G, the number B_1 must be negative if the solution of (4.17) for $\eta > \frac{1}{2}\pi$ is to die out as $\zeta_1 \to \infty$. In the general case, instead of $B_1 \leqslant 0$, we have $a^*(\eta) - \xi_1(\eta) \leqslant 0$, in $\eta \geqslant \alpha$. The eastern boundary current in $\eta > \alpha$ can be treated in precisely the same way and the only new requirement, $a^*(\eta) - \xi_2(\eta) \leqslant 0$, is already implied by (4.18). Similarly, the boundary layer just above $\eta = \alpha$ provides no new criteria.

At this stage, then, we find a family of flow descriptions which has discrepancies in vorticity transport in the corners near $\{\xi_1(\alpha),\alpha\}$, and near $\{\xi_2(\alpha),\alpha\}$. The lack of uniqueness is contained in the function $a(\eta)$ [which is restricted only in that $a(\eta) \geqslant \xi_2(\eta)$ for $\eta < \alpha$] and the function $a^*(\eta)$ [which must only obey the inequality $a^*(\eta) \leqslant \xi_1(\eta)$ for $\eta > \alpha$].

This lack of uniqueness cannot be removed in a purely inertial theory and we must invoke further considerations to partially resolve the difficulties.

5. The effects of turbulent diffusion

It was noted in § 3.3 that the integral of the left side of equation (4.1) over the domain cannot be zero. $g(\eta)$ is a positive function over that domain and the other terms provide a vanishing contribution. It is not surprising, then, that there should be discrepancies in the vorticity transport. One can attempt to resolve this difficulty in either of two ways. The first of these is to replace (4.1) by an equation in which the frictional effects are modelled in some specific way and then solve that equation. If the friction were introduced by a pseudo-laminar model with an eddy viscosity appropriate to the turbulent state of the fluid, equation (4.1) would be replaced by

$$\gamma\Delta\Delta\psi + \psi_\xi(\epsilon\Delta\psi + f)_\eta - \psi_\eta(\epsilon\Delta\psi + f)_\xi + g(\eta) = 0, \tag{5.1}$$

where γ has been defined following equation (3.20). This equation could be solved for very small γ, an appropriate choice since our present knowledge of turbulent transport in the oceans indicates that $\gamma \ll \epsilon$ in the continental boundary regions.

In doing so, we would regard the results of the inertial theory of § 4 as the description of the interior flow of a new boundary-layer problem, in which the viscous terms then provide the mechanism for additional boundary layers along the continental barriers. These frictional boundary layers are much thinner than the inertial layers they adjoin. The detailed solution is of little interest since this friction model is of questionable validity; fortunately, it is also unnecessary. Once we are convinced of the existence of a frictional boundary layer of thickness $\delta \ll \epsilon^{\frac{1}{2}}$ along, say, the western boundary below $\eta = \alpha$, we can evaluate its con-

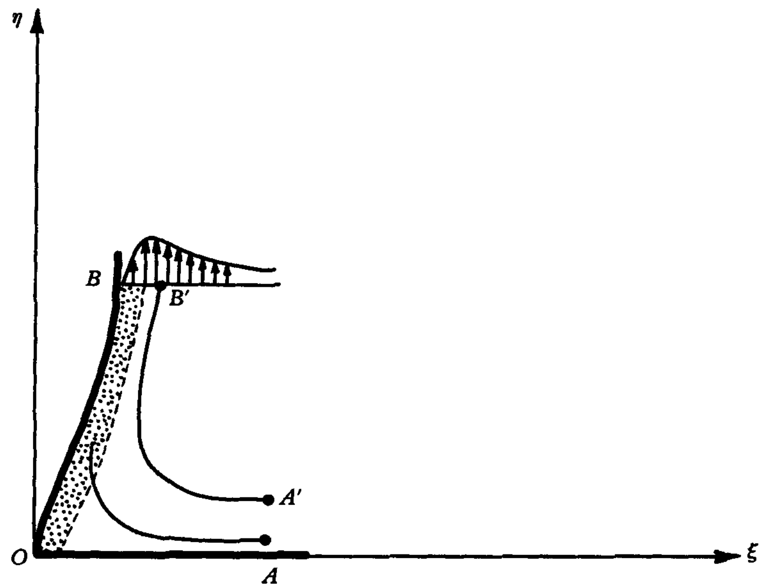

FIGURE 6. Western velocity distribution and friction-layer geometry.

tribution to the gross vorticity balance as follows. Figure 6 shows, schematically, the friction layer and a stream line, $A'B'$, lying entirely outside the friction layer, but near the landward edge of the inertial current.

The convection of vorticity across AA' is negligible in view of the very small velocity ψ_η in this region; the diffusion of vorticity across OA and across $A'B'$ are negligible because each of these streamlines does not lie in the friction layer. Thus, the diffusion of vorticity across OB must be equal to the efflux of vorticity across BB'. The latter quantity, E, is given by

$$E = \int_B^{B'} vv_x dx = \tfrac{1}{2}\{v^2(B') - v^2(B)\} = \tfrac{1}{2}v^2(B'). \tag{5.2}$$

We conclude that the vorticity contributed by diffusion across the barrier via the friction layer OB is equal to one-half the square of the inertial-theory velocity evaluated at the wall. This, as we showed in § 4, is proportional to A_1. We see, by similar calculations, that the (counter-clockwise) vorticity diffused across the upper western boundary is B_1.

The following arguments indicate that negligible vorticity is diffused across the eastern boundary. Note that although the vorticity flux in the horizontal current may lie in a very slender ribbon of fluid at the western end, any such concentration of vorticity will diffuse broadly as the current proceeds towards the eastern boundary. Thus, a very small part of this vorticity will enter the eastern friction layers. Furthermore, little vorticity emerges from the eastern friction layers since the inertial current velocity at $\eta = 0$ and at $\eta = L$ is zero. Since this is a decelerating flow, these friction layers must be less well defined (more diffuse) than those on the western boundary. The vorticity balance corresponding to that illustrated in figure 6 for the western boundary is shown in figure 7. Since little vorticity crosses $D'C'CD$, no important vorticity contributions can diffuse across $D'E'ED$.

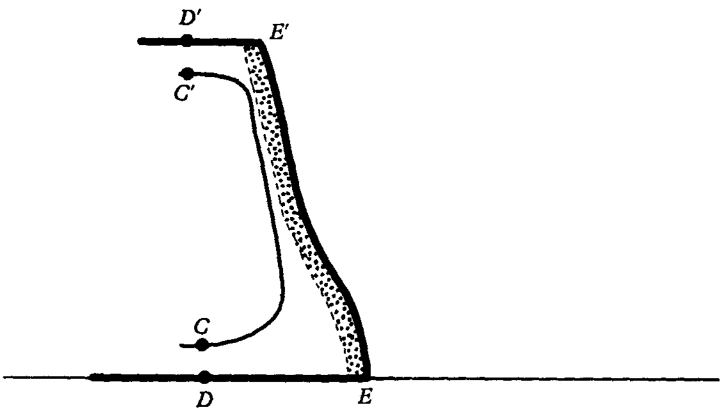

FIGURE 7. Eastern friction layer and geometry.

It follows (once the dimensional parameters have been included properly) that

$$A_1 + B_1 = \int_{\Gamma} g(\eta)\, d\eta\, d\xi, \tag{5.3}$$

where Γ is the domain defined following equation (4.1). Since $A_1 \geqslant 0$, $B_1 \leqslant 0$, the weakest possible circulation is that for which $B_1 = 0$ and

$$A_1 = \int g\, d\eta\, d\xi. \tag{5.4}$$

The degree of difficulty associated with the detailed integration of the foregoing equations depend markedly on the detailed description of $f(\eta)$, $g(\eta)$, $a(\eta)$, $a^*(\eta)$, etc. However, the conclusions are rather insensitive to such detail. If $g(\alpha)$ is unity and if $A_1 = a(\alpha)$, $B_1 = a^*(\alpha)$, the foregoing conclusions still hold. This theory, therefore, gives a lower bound on the circulation which is significantly greater than (by a factor of 2) and in much better agreement with the observations† than is the prediction of previously discussed models.

† That is, the observed transports as reported by Munk (1950, table 2). Since the transports are deduced from observed profiles of density under the assumption of a level of no meridional motion, care must be exercised in their interpretation. It is hoped to be able to remove partially this ambiguity by returning to the data with less restrictive assumptions (Stommel, private communication; see Stommel 1958, p. 164).

6. Discussion

The general consequences of the model can now be described. We consider any wind stress distribution† whose curl has a single maximum at latitude $\eta = \alpha$ and vanishes at the two latitudes $\eta = 0$ and $\eta = L > \alpha$. The basin geometry is that of figure 4. When we postulate that no flow crosses the latitude $\eta = 0, L$, in reasonable conformity with Northern hemisphere observations,‡ the flow over most of the basin [excluding the neighbourhood of $\xi = \xi_1(\eta)$, $\xi = \xi_2(\eta), \eta = \alpha$] is given by equations (4.5) and (4.17). *These descriptions are not unique and cannot be made so without detailed solution of a dissipative model.* However, the constraints imposed by gross dissipative considerations are already very informative.

The intensive flow near $\xi = \xi_1(\eta)$ and $\xi = \xi_2(\eta)$ can be expected to be of boundary-layer type, but these boundary layers cannot be continued from $\eta < \alpha$ into $\eta > a$. It follows from the disjoint character of the layer along ξ_1 (for example) that another intensive current must move eastward from the point $[\xi_1(\alpha), \alpha]$. Note that these inertial boundary currents provide constraints on the general flow described by (4.5) and (4.17), which have the form $a(\eta) > \xi_2(\eta)$ in $\eta < \alpha, a^*(\eta) < \xi_1(\eta)$ in $\eta > \alpha$. The gross vorticity balance further restricts the choice of $a(\eta)$ and $a^*(\eta)$ by demanding that

$$a(\alpha) + a^*(\alpha) = \int g(\eta)\, d\eta\, d\xi.$$

Since $a^* \leqslant \xi_1(\eta)$, the smallest value for $a(\alpha)$ is given by

$$a(\alpha) > \int g(\eta)\, d\xi\, d\eta - \xi_1(\alpha). \tag{6.1}$$

This provides a lower bound for $a(\alpha)$ which supersedes the foregoing constraints $a(\alpha) \geqslant \xi_2(\alpha)$.

Using, now, any $a(\eta)$ and $a^*(\eta)$ which obey the foregoing constraints, we find in the region $\eta < \alpha$ that V, the net vorticity input from $g(\eta)$ and from solid boundaries, is not zero but negative and that the input in the region $\eta > \alpha$ is $-V$.

The current along $\eta = \alpha$ must therefore accomplish two things. It must transfer from the upper half-basin into the lower half-basin all of the vorticity which diffused into the stream along $\xi = \xi_1(\eta)$, $\eta > \alpha$ and that which was put in by the wind-stress curl, $g(\eta)$. It must also change the distribution of vorticity among streamlines in such a way that fluid enters the interior region of the lower basin with a vorticity commensurate with $\psi^{(0)}$. To do this, the width of the zone affected by friction (turbulent transfer) must be as wide as the inertial current.

† Despite the introduction of our model as an effort to understand primarily the circulation structure in the 15° to 55° N. latitude range, it is equally appropriate to the lower latitudes where similar but narrower latitude bands exist with wind-stress curl variations from zero to a maximum to zero. In these latitude bands, the complex subsurface current structure seems to be consistent with this analysis and the β-plane approximation ($f \simeq f + \beta\eta$) is a very good one indeed. (We are, of course, not referring to the Cromwell current.)

‡ The geometry is much more complicated in the Southern hemisphere and will be discussed elsewhere.

As a dominant process, we suggest the instability associated with the fact that the fluid enters the mid-latitude jet with velocities which are of order $\epsilon^{-\frac{1}{2}}$, instead of the $\epsilon^{-\frac{1}{3}}$ appropriate to such a stream;† the entering vorticity is correspondingly high. The deceleration process is certain to be violent and the transfer occurs. It is clear, however, that this argument as it stands is not adequate to provide a mechanistic description of the transfer process or a quantitative prediction of $a(\alpha)$.

The foregoing combination of an inviscid boundary-layer theory and a heuristic dissipation model gives a self-consistent description of ocean-basin flows which are in rather good agreement with Northern hemisphere observations and at least suggestive for the Southern hemisphere. In particular, the quantity $a(\alpha)$, which defines the transport, is required by this theory to be twice as large as that of previous theories. The minimum values acceptable within this theory are close to those of the real oceans.

In brief, then, we *deduce* the presence of the mid-latitude jet and an estimate of the circulation; we *infer* the gross character and location of the turbulent vorticity by merely noting the role of the diffusive processes which are required in order that they together with the clearly deduced flow form a coherent and self-consistent physical picture.

It is a pleasure to acknowledge the stimulation afforded by Professor Henry Stommel's continued interest. Thanks are due to the Office of Naval Research (GFC) and to the Research Corporation (ARR) for their support during the period of this research.

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† More precisely, the fluid must enter the corner region with velocity of one amplitude and leave with velocity of another amplitude. The resulting instability may enlarge the transition region, which conceivably includes the whole region of maximum meandering.

Note added in proof

As was pointed out to the authors by Derek Moore, the reader could infer, in §3.3 and in §4, that one can construct by conventional laminar boundary-layer analysis a description of a frictional modification of the inertial flow pattern derived earlier. This is certainly correct in so far as the western boundary regions are concerned, but it is not true for the eastern boundaries. The decelerating character of the inviscid flow near the eastern boundaries implies that such conventional boundary layers cannot be constructed. This is disturbing only because one cannot state with assured accuracy the distance to which boundary-generated vorticity will migrate. It is unlikely that the general flow pattern will be modified appreciably by this difficulty, and especially unlikely that the circulation-estimating inequality (which is controlled by western boundary considerations) will be affected. In other words, a constant eddy coefficient as introduced is inappropriate for a discussion of the structure of the flow field, although it is useful for some order of magnitude estimates.